

Phase estimation with two-mode squeezed vacuum and parity detection: Bayesian analysis

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Abstract. A recently proposed phase-estimation protocol that is based on measuring the parity of a two-mode squeezed vacuum state at the output of a Mach-Zehnder interferometer shows that sub-Heisenberg sensitivity is obtained (PM Anisimov et al, PRL 104, 103602 2010). This protocol works in the case of infinite number of parity measurements. Here we consider the use of photon-number-resolving detectors to measure parity and we use Bayesian analysis to characterize the performance of the phase estimation in such a scheme but with finite number of measurements. We have found that phase estimation has non-zero bias at 0 or $\pi/2$ phase. Yet there is an in-between region where the bias is negligible. Our phase estimation scheme beats the Heisenberg limit and saturates the Cramer-Rao bound.

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1. Introduction

Phase estimation is the primary tool for optical metrology. There are several experimental ways to estimate phase. Coherent light based interferometry is most commonly used but its sensitivity for phase estimation is limited by the shot-noise (SN) [1]. This is not a problem in the case of limitless resources or in the case of samples that can withstand large doses of radiation. However, this is a problem otherwise, and one has to resort to interferometry with quantum states of light, such as N00N states [2], and measuring parity [3, 4] in order to achieve sub-shot-noise or even Heisenberg limited (HL) sensitivity of phase estimation.

A recently proposed phase estimation scheme beats the HL in the case of infinite number of parity measurements [6]. This scheme measures the parity of the output signal in a Mach-Zehnder interferometer (MZI), Fig. 1, with two-mode squeezed vacuum (TMSV) input. It turned out that this particular scheme using TMSV input has sub-Heisenberg sensitivity even with linear phase evolution. This is due to the fact that the photon number uncertainty is greater than the mean photon number inside the MZI.

Squeezed vacuum generation in an optical parametric amplifier (OPA) of up to 12 dB of quadrature squeezing has been achieved experimentally [7]. This corresponds to a mean photon number in both modes of the TMSV of up to seven photons. Hence, the parity, even or odd photon number, of the state can be measured with existing photon-number-resolving detectors such as transition edge sensors [8, 9]. However, a photon-number-resolving detector does not provide a mean value of the parity signal after a single measurement, which means that a phase measuring experiment must be repeated multiple times.

Our paper applies Bayesian analysis to this phase-estimation scheme when photon-number-resolving detectors are used to infer the parity signal. We use the parity signal at the output of the MZI to estimate the unknown phase θ . Our analysis shows that, although phase estimation is biased near the phase origin and at $\pi/2$, there is an in-between interval where unbiased phase estimation is possible. In this interval, phase sensitivity saturates the Cramer-Rao bound (CRB), beats the HL, and remains below the SN limit. Unlike the recently proposed phase estimation scheme, we achieve these results with a finite number of parity measurements.

2. Model

Parity based phase estimation was originally introduced to quantum optics by Gerry in Ref. [10]. The parity signal $\langle \hat{\Pi} \rangle$ for our phase estimation scheme,

$$\langle \hat{\Pi} \rangle = \frac{1}{\sqrt{1 + \bar{n}(\bar{n} + 2) \sin^2 \theta}}, \quad (1)$$

was obtained in Ref. [6]. It depends on the unknown phase θ inside of the MZI and the mean photon \bar{n} in the TMSV state used. Thus, knowing \bar{n} and $\langle \hat{\Pi} \rangle$, the unknown phase θ can be determined.

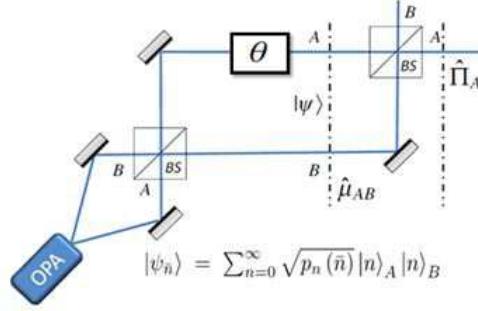


Figure 1. A schematic view of a Mach-Zehnder interferometer (MZI). Two-mode squeezed vacuum states are generated at the input of the MZI by an optical parametric amplifier. We measure the parity signal at the output of the MZI with a photon number resolving detector such as a transition edge sensor.

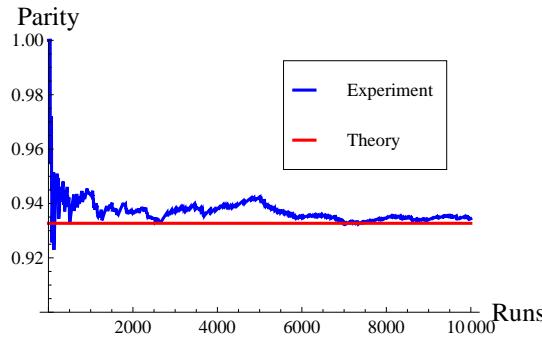


Figure 2. Convergence of the parity signal (blue) to its expectation value of $\langle \hat{\Pi} \rangle = 0.93$ (red) in a finite number of $M = 10^4$ numerical runs. A different sample of this convergence will evolve differently due to the probabilistic nature of the scheme, but it will always converge to its expectation value. In this case the photon number is $\bar{n} = 3$ and the unknown phase is $\theta = 0.1$.

Each parity measurement returns either an even or odd outcome with probabilities P_e and P_o . Thus, there is uncertainty in measuring the parity signal for our phase estimation scheme, so it requires averaging the outcomes of many parity measurements. The expectation value of a state's parity is $\langle \hat{\Pi} \rangle = 1 \cdot P_e + (-1) \cdot P_o$. Since, $P_e + P_o = 1$, we can take advantage of the following expressions for the probabilities of an even and an odd outcome respectively:

$$P_e = \frac{1}{2} \left(1 + \langle \hat{\Pi} \rangle \right) \quad (2)$$

$$P_o = \frac{1}{2} \left(1 - \langle \hat{\Pi} \rangle \right) \quad (3)$$

We use the probability of an even photon number, P_e from Eq. (2), to numerically generate a value for the parity signal. Figure 2 shows how the parity signal (blue) converges to its expectation value of $\langle \hat{\Pi} \rangle = 0.93$ (red) in a set of $M = 10^4$ numerical runs for the case of an input photon number of $\bar{n} = 3$ and an unknown phase $\theta = 0.1$. Repeating this procedure with the same parameters results in a slightly different parity evolution due to the probabilistic nature of the scheme. The uncertainty of the signal

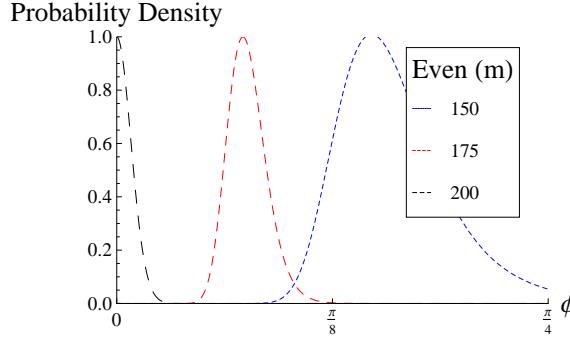


Figure 3. The Bayesian probability density function (PDF) from Eq. (5) is evaluated with mean photon number $\bar{n} = 3$ and numerical runs $M = 200$ for various number of outcomes m . We rescaled the heights for easier comparison of their widths and maxima. As the number of even outcomes decreases from $m = M$ to $m = 0.75M$, the maximum of the PDF shifts towards $\phi = \pi/2$ with a consequent broadening of the distribution. This broadening reduces the accuracy of estimating phase.

leads to an uncertainty in the measurement of the unknown phase θ . We can improve the certainty of unknown phase estimation by repeating our measurement many times so that it converges to the expectation value.

The Bayesian analysis of an even or odd parity detection at the output of the MZI provides a probability for the unknown phase θ to be in the interval $[\phi, \phi + d\phi]$ given all prior observations. This analysis is based on the update rule for the probability density function (PDF) given by Bayes theorem: $P(\phi|\text{output}) \propto P(\text{output}|\phi)P_{\text{prior}}(\phi)$, so that each consecutive observation modifies the PDF and improves phase estimation. Hence, the update equation given by Bayes theorem for an even or odd outcome is:

$$P(\phi|\{e, o\}) \propto P(\{e, o\}|\phi)P(\phi). \quad (4)$$

Particular outcomes in our numerical model are independent thus the Bayesian PDF after M runs with m even outcomes is:

$$P(\phi|m) \propto P_e^m(\phi)P_o^{M-m}(\phi). \quad (5)$$

Figure 3 presents use of this update rule in the case of $M = 200$ runs with $\bar{n} = 3$ input photons for different number of even outcomes: $m = 150, 175, 200$. The heights of each have been rescaled for the ease of comparison.

One can see two effects that reduced number of even outcomes has on the PDF. The PDF becomes broader and shifts its maximum towards $\phi = \pi/2$. This results in higher phase uncertainty and thus reduction in the sensitivity of phase estimation near $\theta = \pi/2$. This coincides with predictions of Ref. [6] that the best phase sensitivity is achieved in the vicinity of $\theta = 0$, where the parity of the output state is predominantly even.

As M goes to infinity the PDF becomes

$$P(\phi, \theta, M) \propto [P_e^{P_e(\theta, \bar{n})}(\phi, \bar{n})P_o^{P_o(\theta, \bar{n})}(\phi, \bar{n})]^M. \quad (6)$$

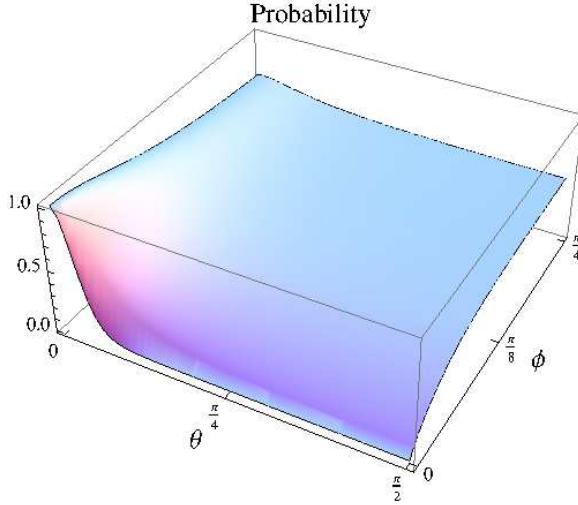


Figure 4. Core phase dependence of Bayesian PDF for $\bar{n} = 3$ in the limit of infinitely long averaging. A well defined maximum is present only near expected phase $\phi = 0$ and our unknown phase $\theta = 0$ which explains good sensitivity at the vicinity of the phase origin.

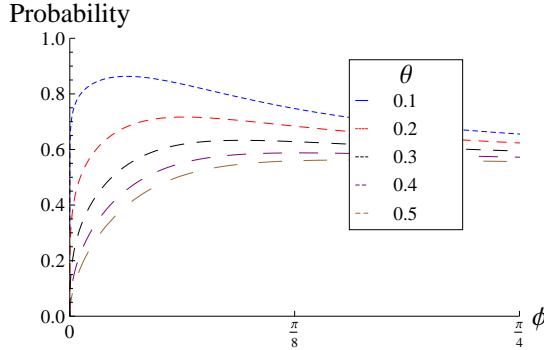


Figure 5. Slices of the theoretical probability density obtained with Bayesian analysis over long averaging. We show various unknown phase's θ with an input photon number of $\bar{n} = 3$. Low values of θ have well defined maximum, which provides a scheme for the fast and accurate estimation of the unknown phase θ .

We use a phase ϕ that maximizes the PDF as our maximum likelihood estimator for an unknown phase θ . Success of this approach is governed by a core function $P_e^{P_e(\theta, \bar{n})}(\phi, \bar{n})P_o^{P_o(\theta, \bar{n})}(\phi, \bar{n})$ that is presented in Fig. 4 as a function of two phases, θ and ϕ , for $\bar{n} = 3$. Figure 5 shows slices of this core function. Only slices of the core function for low values of the unknown phase θ demonstrate a well-defined maximum. Larger values of θ yield ill defined maxima causing reduced accuracy and slow convergence of our scheme. A well defined maximum provides a scheme for the fast and accurate estimation of the unknown phase θ .

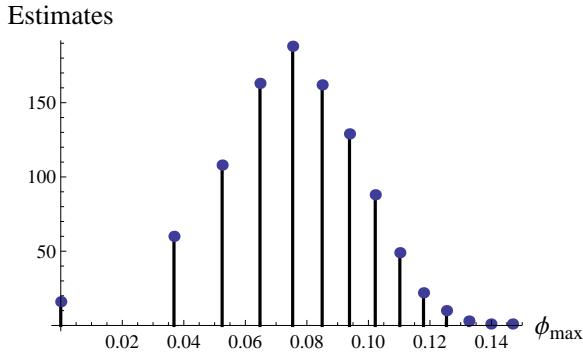


Figure 6. This distribution of phase estimations in 1000 trials shows uncertainty of phase estimation due to a finite number of numerical runs M for $\theta = 0.08$ and $\bar{n} = 3$. Each phase estimation is obtained by inputting the number of even states m counted in $M = 200$ numerical runs into Eq. (5) and then finding its maximum. This distribution has the mean value of $\bar{\phi} = 0.078$ and a standard deviation of $\Delta\bar{\phi} = 0.022$ which falls well within the unknown phase of $\theta = 0.08$.

3. Numerical results

Equipped with theoretical predictions, we now implement our phase estimation scheme numerically assuming that parity signal is measured with photon-number resolving detectors that detect even outcomes with the following probability:

$$P_e = \frac{1}{2} \left(1 + \frac{1}{\sqrt{1 + \bar{n}(\bar{n} + 2) \sin^2 \theta}} \right) \quad (7)$$

This comes directly from Eq. (2) which depends on an unknown phase θ and input photon number \bar{n} . In a single phase estimation, we repeat the process of outcome generation M times and determine a particular number of even outcomes. Using our Bayesian update rule from Eq. (5) with m even outcomes, we calculate the probability distribution for an unknown phase θ . Our maximum likelihood estimator concludes that the unknown phase is equal to the phase at the maximum of this probability distribution.

Normally, phase sensitivity is estimated by measuring the width of a Bayesian PDF. In our case however, the Bayesian PDF is asymmetric and has several maxima in a 2π interval. Therefore, we analyse the statistics of observed maxima in the interval $\phi \in [0, \pi/2]$ for a phase estimation instead. To determine uncertainty of our phase estimation, we repeat phase estimation 1000 times with $M = 200$ numerical runs in each. Figure 6 shows the distribution of different phase estimations in the case of unknown phase being $\theta = 0.08$ and $\bar{n} = 3$. The distribution of phase estimations have a mean value of $\bar{\phi} = 0.078$ with a standard deviation of $\Delta\bar{\phi} = 0.022$ which falls well within the expected phase of $\theta = 0.08$.

As we ran our phase estimator with input intensity $\bar{n} = 3$ for several unknown phases in the interval $\theta \in [0, \pi/2]$, we found the presence of bias, which is statistical favouritism that causes misleading results, defined as $Bias = \bar{\phi} - \theta$.

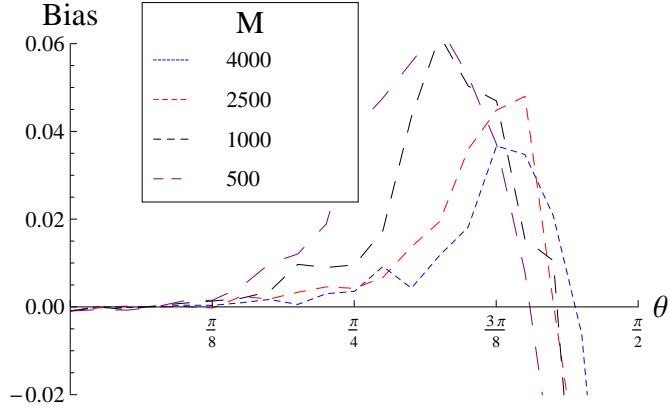


Figure 7. Bias of phase estimation for $\bar{n} = 3$ as a function of the unknown phase θ for increasing number of numerical runs M and 1000 trials in each phase estimation. Bias is negligible in the interval $\theta \in [0.08, 0.6]$, and thus provides us with an accurate phase estimation scheme.

Figure 7 demonstrates high bias in estimating the unknown phase near the boundaries of the interval. This is shown for a various number of numerical runs M which demonstrates how more measurements improve our scheme. For an unknown phase $\theta \in [0, 0.08]$, our phase estimation exhibits a large bias toward zero. Likewise, when the unknown phase approaches $\pi/2$ our scheme again becomes biased. This leaves an optimal phase interval of $\theta \in [0.08, 0.6]$ that offers unbiased phase estimation.

The interval for unbiased phase estimation depends on the input intensity \bar{n} . For a larger $\bar{n} = 7$, our phase estimator shows that the minimum value of the unbiased phase interval remains relatively unchanged; however, the length of the interval reduces due to signal localization near the phase origin. Namely, the right boundary of the interval reduces from $\theta = 0.6$ to $\theta = 0.3$.

4. Phase sensitivity

The uncertainty of phase estimation $\Delta\phi$ quantifies the phase sensitivity of the scheme. In the case of conventional phase estimation with coherent laser light and intensity difference measurement, phase sensitivity is shot-noise limited, $\Delta\phi > 1/\sqrt{M\bar{n}}$. This sensitivity improves with increasing input intensity as well as the number of numerical runs. However, ultimate phase sensitivity is Heisenberg limited, $\Delta\phi > 1/\sqrt{M\bar{n}^2}$, that has the same dependence on the number of numerical runs M but faster dependence on input intensity.

Phase estimation with TMSV and parity detection is capable of beating the HL due to a greater photon number variance in the state. Phase sensitivity for this scheme is limited from below by the CRB [6]:

$$\Delta\phi = \frac{1 + \bar{n}(2 + \bar{n}) \sin^2 \theta}{\sqrt{M\bar{n}(2 + \bar{n})} \cos \theta}. \quad (8)$$

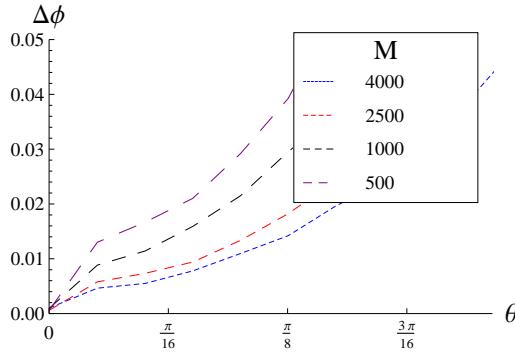


Figure 8. The standard deviation of phase estimation with an input photon number of $\bar{n} = 3$ for increasing number of numerical runs M . We found that more phase estimates reduces the standard deviation. As θ increases towards $\pi/2$ the standard deviation becomes large. This explains why bias is high when $\theta > 0.6$.

Hence, an optimum phase sensitivity, $\Delta\phi = 1/\sqrt{M\bar{n}(2 + \bar{n})}$, obtained near $\theta = 0$ is sub-Heisenberg.

Our presented approach of measuring parity with photon-number-resolving detectors is however biased near $\theta = 0$. Thus, Bayesian phase estimation with photon-number resolving detectors can only be obtained in the low bias interval of $\theta \in [0.08, 0.6]$. Figure 8 shows the standard deviation, $\Delta\bar{\phi}$, of our phase estimation scheme. Increasing the number of runs in a phase estimate improves the phase sensitivity when θ is near the origin. This is because phase uncertainty scales as $\Delta\phi = c/\sqrt{M}$, where the proportionality constant c depends on the unknown phase θ and input photon number \bar{n} . As we calculate the unknown phase θ in various regions of bias with $\bar{n} = 3$, we find that the standard deviation is low near the origin.

As an example, consider $\theta = 0.01$ which is in a region of high bias. As we increase the number of numerical runs in a phase estimate, we find that the standard deviation remains constant and cannot be fitted to a c/\sqrt{M} function in order to determine the proportionality coefficient c (see Fig. 9). The same behavior is observed in the other high-bias interval closer to $\pi/2$.

In the following, we focus on unknown phases in the interval suitable for unbiased phase estimation with parity measurement using photon number resolving detectors. Our data does have the expected $1/\sqrt{M}$ dependence. Hence, we can compare the proportionality coefficient obtained from the fitting procedure with a value expected from the CRB as well as from the SN limit.

Figure 10 presents the standard deviation of phase estimation as a function of the number of phase estimates for an unknown phase of $\theta = 0.3$ and $\bar{n} = 3$. In this case, we find that the proportionality coefficient is $c_{\text{TMSV}} = 0.69$. As we compare this proportionality coefficient with the SN limit, $c_{\text{SN}} = 0.58$, and the CRB, $c_{\text{CRB}} = 0.62$, obtained from Eq. (8), we see that the sensitivity is in close proximity to the CRB (red). With an unknown phase of $\theta = 0.08$ located closer to the phase origin in our unbiased interval we can improve this result. For a given unknown phase, the sensitivity of

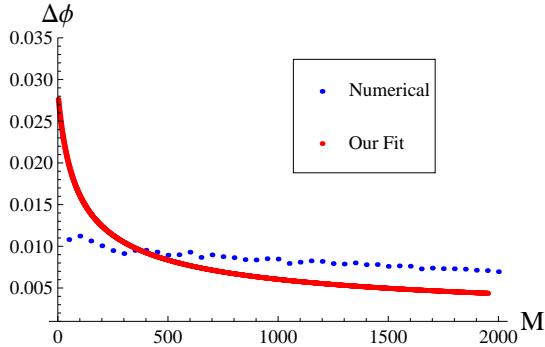


Figure 9. For a single phase estimate we see how the standard deviation changes for increasing numerical runs M with $\bar{n} = 3$ and unknown phase $\theta = 0.01$. We compare our results with that of the Cramer-Rao bound (CRB) and shot-noise (SN) limits. Since $\theta = 0.01$ is in a high biased region, the data does not have the $1/\sqrt{M}$ dependence. We can obtain this dependence with the use of an unknown phase in the unbiased region as shown in Fig. 10.

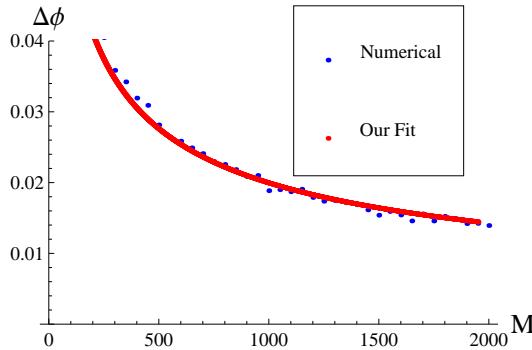


Figure 10. The standard deviation versus the number of numerical runs M in the case of unknown phase $\theta = 0.3$ and photon number $\bar{n} = 3$. It exhibits the expected $1/\sqrt{M}$ scaling with a proportionality coefficient, $c_{\text{TMSV}} = 0.69$. It does not beat either $c_{\text{CRB}} = 0.62$ or $c_{\text{SN}} = 1/\sqrt{\bar{n}} = 0.58$, but we can use an unknown phase of $\theta = 0.08$ which is at the lower end of our unbiased interval to significantly improve this result.

phase estimation could be improved by varying the mean photon number in the input TMSV state. This is analysed for $\theta = 0.3$ and the results are presented in Fig. 11, where the phase sensitivity is characterized by c_{TMSV} which removes the dependence on M and focuses on input intensity dependence. One can see that our previous choice of $\bar{n} = 3$ is an optimum input intensity for this value of the unknown phase. For lower input intensities, phase sensitivity is in close proximity to the CRB. As the input intensity increases, the standard deviation of the current scheme becomes much worse than predicted by the CRB. Therefore, we investigate lower unknown phases in our unbiased interval. The best sensitivity of our phase estimation protocol is expected near the phase origin. Performing phase estimation in the unbiased interval offers significant improvement for phase sensitivity. With an unknown phase of $\theta = 0.08$ shown in Fig. 12, one can see that phase sensitivity of our scheme is nearly optimal and beats

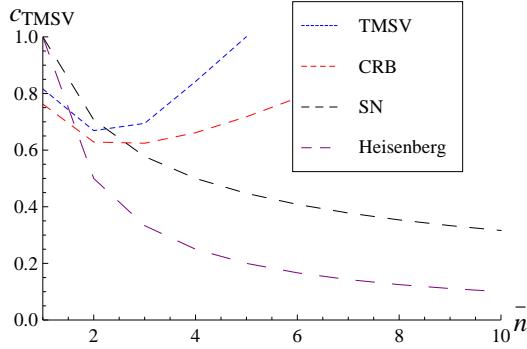


Figure 11. The constant of proportionality, c_{TMSV} , versus the number of photons in the input state for an unknown phase of $\theta = 0.3$. The case of $\bar{n} = 3$ has the smallest standard deviation for phase estimation based on the scheme presented here (blue). The demonstrated sensitivity is in close proximity to the CRB in red. The SN limit in black and the Heisenberg limit (HL) in purple are present for comparison.

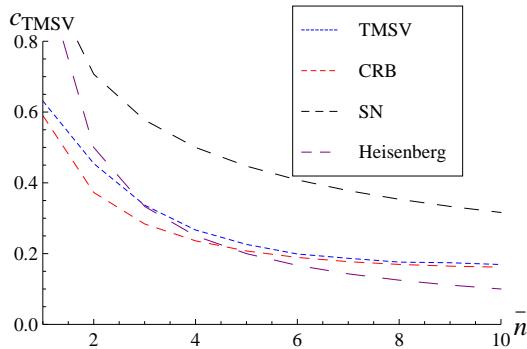


Figure 12. The constant of proportionality, c_{TMSV} , versus the number of photons in the input state for an unknown phase of $\theta = 0.08$. Our numerical experiment (blue) does much better than SN (Black), almost as well as the CRB (red), and does as well as the HL (purple).

the SN limit for an extended range of input intensities. Thus, parity measurement with photon-number-resolving detectors is close to the limiting performance for our scheme and could be used to demonstrate sub-Heisenberg limited phase estimation.

5. Conclusion

Phase estimation protocols that rely on parity measurement benefit from photon number-resolving-detectors that can be used to infer parity of the state. Here, we considered a particular phase estimation protocol that is based on TMSV input and parity detection at the output of the MZI. Use of photon-number-resolving detectors means that measurements must be repeated multiple times with Bayesian analysis applied to all outcomes.

Our scheme shows that phase sensitivity saturates the Cramer-Rao bound and beats the Heisenberg limit with the use of a finite number of experimental runs. We

discovered that our maximum likelihood estimator is biased near the phase origin where the best sensitivity is expected. As a result, the standard deviation of the estimated values does not reduce with increasing number of phase estimations but stays constant. However, small values of the unknown phase in our unbiased interval $\theta \in [0.08, 0.6]$ allow for unbiased phase estimation. Phase sensitivity behaves as expected and remains in close proximity to the Cramer-Rao bound. Consequently, the phase sensitivity remains sub-shot-noise limited for a broad range of input intensities as long as phase estimation is performed near the origin.

Acknowledgments

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6. References

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